Lecture 32

From last time

$\boxed{\text{e.g.}}$

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} = S_\beta \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} S_\beta^{-1}$$

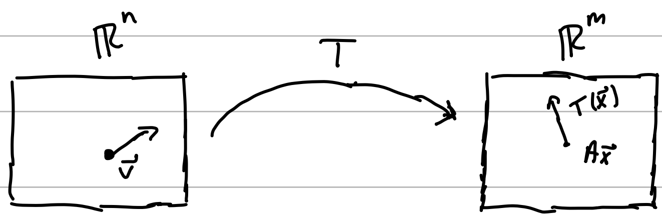
$$\beta = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$$

$$S_\beta = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

New Stuff!

- Only interested in $T: \mathbb{R}^n \to \mathbb{R}^n$ $\quad A \in M_{n\times n}$

  $T(\vec{x}) = A\vec{x}$



<u>Def</u>: Given $T: \mathbb{R}^n \to \mathbb{R}^n$ (linear transformation), $T(\vec{x}) = A\vec{x}$ for some $A \in M_{n\times n}$ we say that $\vec{v} \in \mathbb{R}^n$ is an eigenvector for T (or A) with eigenvalue $\lambda \in \mathbb{R}$ if $T(\vec{v}) = \lambda\vec{v}$ (i.e. $A\vec{v} = \lambda\vec{v}$)

A non zero vector $\vec{v}$

• $A = S_\beta \Lambda S^{-1}$

$\rightarrow A^k = S_\beta \Lambda^k S^{-1} = S_\beta \begin{bmatrix} \lambda_1^k & & 0 \\ 0 & \ddots & \lambda_n^k \end{bmatrix} S_\beta^{-1}$

$\boxed{Ex}$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is eigen vec with eigen value 1

$$\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1}$$

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is eigenvec with eigvalue 2

• How to find evals/evecs of a matrix A?

<u>Obs</u>: If you know evecs it's easier to find e-vals (+ vice versa)

i.e. Given $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, compute $\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

↳ e-vec with e-val 2

• <u>Alt</u>: given $\lambda = 2$, how to find $\vec{v}$?

Seek solution $\vec{v}$ of $A\vec{v} = \lambda \vec{v} = \lambda I \vec{v} \leftrightarrow A\vec{v} - \lambda I \vec{v} = 0$

$\leftrightarrow \vec{v} \in \operatorname{Ker}(A - \lambda I)$

• So if $\lambda = 2$, $\vec{v} \in \operatorname{Ker} \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}$

↳ So solve $\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 1 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \vec{v} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

• So $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigen vector.

Q How to find the eigen vals without knowing e-vecs ?

Prop: Given $A \in M_{n\times n}$ and a scaler $\lambda \in \mathbb{R}$, The following are equivalent:

- $\ker(A-\lambda I)$ is non-trivial
- λ is an e-val of A
- ★ $\det(A-\lambda I)=0$

$\boxed{\text{Ex}}$

$$A-\lambda I=\begin{bmatrix} 3-\lambda & -2 \\ 1 & -\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = (-\lambda)(3-\lambda)-(-2)$$
$$= \underbrace{\lambda^2-3\lambda+2}_{\text{Characteristic polynomial}} = 0$$

$$\to \quad (\lambda-2)(\lambda-1)=0$$
$$\lambda=2, \quad \lambda=1$$

- Now that you know the eigen-vals you can find eigen-vecs with process above.

From last time

$\boxed{\text{Eg}}$ $T: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad T(\vec{x}) = \vec{x}$ rotated ↺ by $\frac{\pi}{2}$ has no (real) eigenvectors.

$$[T] = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = A$$

e-vals are solns of $\det(A - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0$

$$\lambda^2 + 1 = 0$$

$\to \lambda^2 = -1$

$\to \lambda = \pm i$!!! (not good)

Moral: Not all linear transformations $T: \mathbb{R}^n \to \mathbb{R}^n$ are diagonalizable.

$\boxed{\text{E.g}}$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\det(A - \lambda I) = \det(I - \lambda I) = \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)^2$$

$$(1-\lambda)^2 = 0$$

$$\lambda = 1, 1$$

$[A - \lambda I \mid \vec{0}]$

$[I - I \mid \vec{0}]$

$[0 \mid \vec{0}]$

$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \to \vec{v} = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

- you can have independent e-vecs for same e-val.

$\boxed{\text{Ex}}$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

e-vals: $0 = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix}$

$$0 = (1-\lambda)^2 - 0$$

$$\lambda = 1, 1$$

- now to find e-vecs must solve $[A - I \mid \vec{0}]$

$$\rightarrow \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \vec{v} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- so $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable.